A Game Theoretic Analysis of Intrusion Detection in Access Control Systems[1]- **Tansu Alpcan & Tamer Başar**

FEP3301 Computational Game Theory
Paper Presentation

Kim Hammar

kimham@kth.se

Division of Network and Systems Engineering
KTH Royal Institute of Technology

Oct 14, 2021

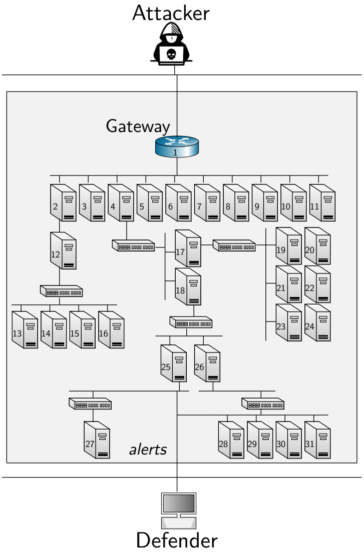
[1] T. Alpcan and T. Basar. "A game theoretic analysis of intrusion detection in access control systems". In: *2004 43rd IEEE Conference on Decision and Control (CDC) (IEEE Cat. No.04CH37601)*. Vol. 2. 2004, 1568–1573 Vol.2. DOI: 10.1109/CDC.2004.1430267.

Outline

- Use case: **Intrusion Detection**
- System model
- A finite **extensive-form** model of the use case
  - Elementary equilibrium analysis
- A static **continuous-kernel game** model of the use case
  - Equilibrium analysis using Rosen's Theorem

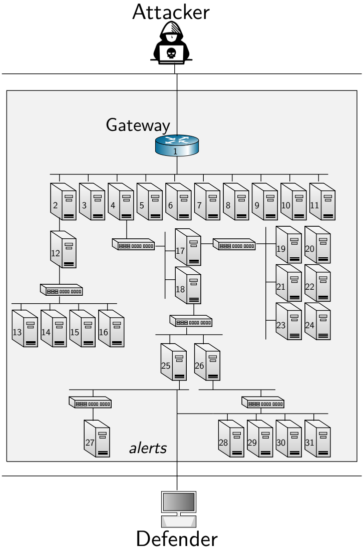
Use Case: Intrusion Detection

- **Defender = IDS**:
  - System Operator with task of detecting intrusions
- **IT Infrastructure**:
  - Consist of a set of components (also called “subsystems”).
  - The infrastructure is equipped with a network of sensors
  - Sensors report anomalies/alerts to the defender
  - Each component has a set of vulnerabilities
- **Attacker**:
  - Can attack vulnerable components



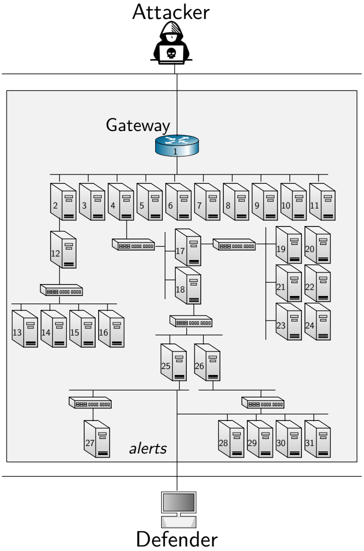
Use Case: Intrusion Detection

- **Defender = IDS**:
  - System Operator with task of detecting intrusions
- **IT Infrastructure**:
  - Consist of a set of components (also called “subsystems”).
  - The infrastructure is equipped with a network of sensors
  - Sensors report anomalies/alerts to the defender
  - Each component has a set of vulnerabilities
- **Attacker**:
  - Can attack vulnerable components



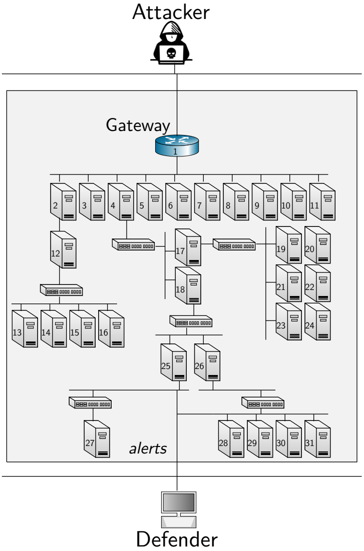
Use Case: Intrusion Detection

- **Defender = IDS**:
  - System Operator with task of detecting intrusions
- **IT Infrastructure**:
  - Consist of a set of components (also called “subsystems”).
  - The infrastructure is equipped with a network of sensors
  - Sensors report anomalies/alerts to the defender
  - Each component has a set of vulnerabilities
- **Attacker**:
  - Can attack vulnerable components



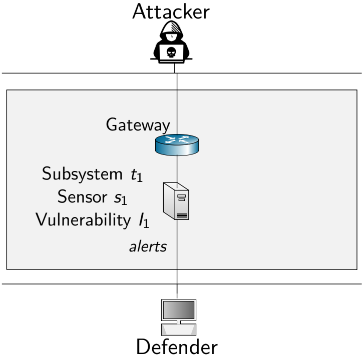
System Model

- **Infrastructure components (Subsystems)**
  - $\mathcal{T} = \{t_1, t_2, \ldots t_{max}\}$
- **Vulnerabilities**
  - $\mathcal{I} = \{I_1, I_2, \ldots, I_{max}\}$
- **Attacks**
  - $\mathcal{A} = \mathcal{T} \times \mathcal{I}$
- **Sensors**
  - $\mathcal{S} = \{s_1, s_2, \ldots, s_{max}\}$
  - Each sensor reports “alarm” or “no alarm”.
  - Each sensor $s_i \in \mathcal{S}$ corresponds to a vulnerabilitiy $I_i \in \mathcal{I}$

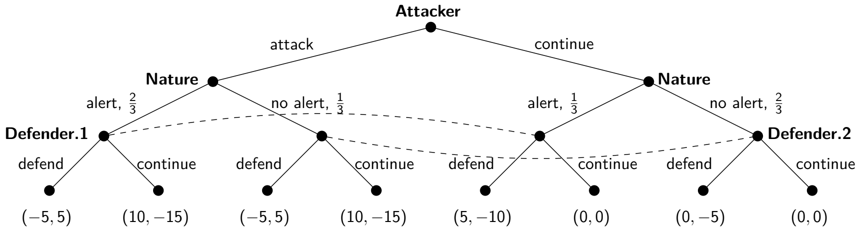


Example Infrastructure: One Subsystem, One Sensor and One Vulnerability

- **Infrastructure components (Subsystems)**
  - $\mathcal{T} = \{t_1\}$
- **Vulnerabilities**
  - $\mathcal{I} = \{l_1\}$
- **Attacks**
  - $\mathcal{A} = \{a = (t_1, l_1)\}$
- **Sensors**
  - $\mathcal{S} = \{s_1\}$

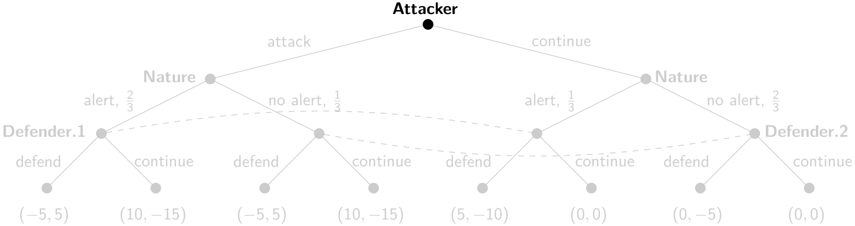


Finite Extensive-Form Game Model

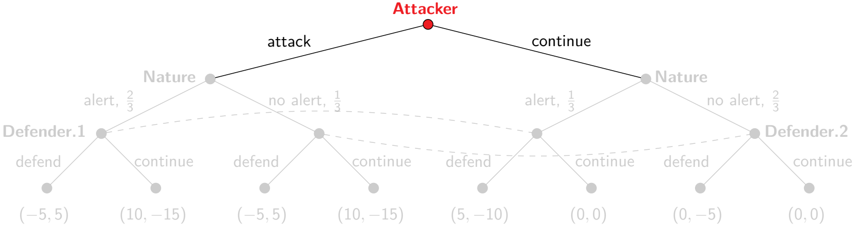


- **Players**: $N = \{\text{Attacker}, \text{Defender}, \text{Nature}\} = \{1, 2, 3\}$
- **Action sets**:
  - $A_1 = \{\text{Attack}, \text{Continue}\}$, $A_2 = \{\text{Alert} \rightarrow \text{Defend}, \text{Alert} \rightarrow \text{Continue}, \text{No Alert} \rightarrow \text{Defender}, \text{No Alert} \rightarrow \text{Continue}, \}$, $A_3 = \{\text{Alert}, \text{No Alert}\}$
- **Nature's pre-defined strategy:**
  - $f(\text{Alert}|\text{Attack}) = \frac{2}{3}$, $f(\text{Alert}|\text{Continue}) = \frac{1}{3}$

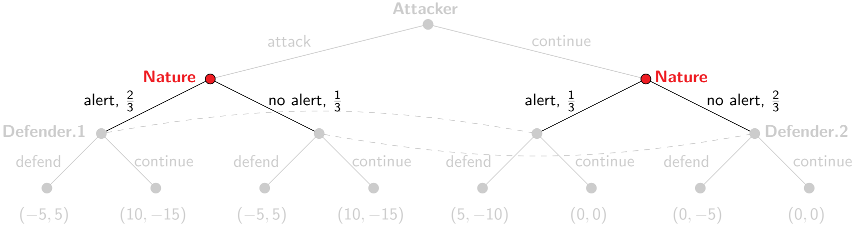
Finite Extensive-Form Game Model



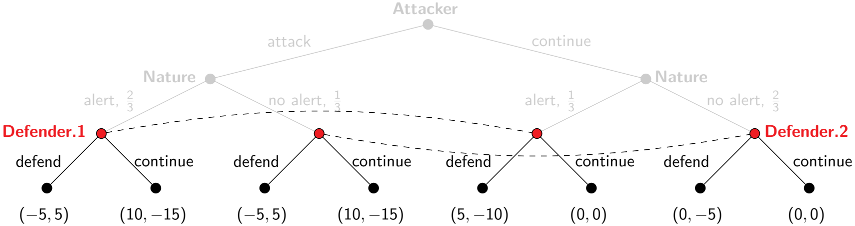
Finite Extensive-Form Game Model



Finite Extensive-Form Game Model

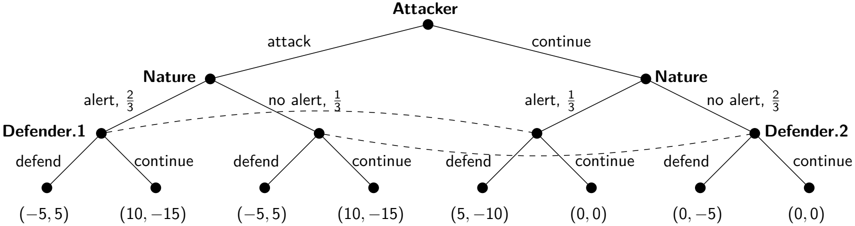


Finite Extensive-Form Game Model



Extensive & Strategic Form of the Finite Game Model

Extensive Form:

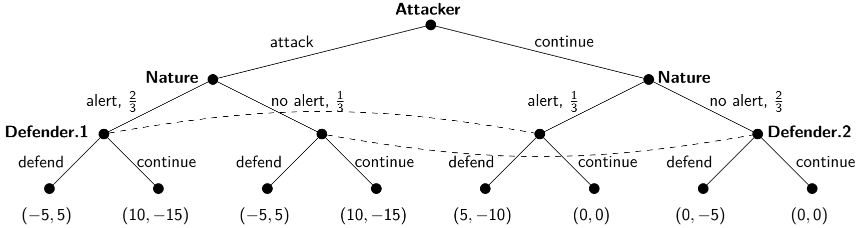


Strategic Form:

$$
\begin{array}{c} \\ A \\ C \end{array}
\begin{array}{c}
\begin{array}{cccc} DC & DD & CD & CC \end{array} \\
\left[\begin{array}{cccc}
0, -\frac{5}{3} & -5, 5 & 5, -\frac{25}{3} & 10, -15 \\
\frac{5}{3}, -\frac{10}{3} & \frac{5}{3}, -\frac{20}{3} & 0, -\frac{10}{3} & 0, 0
\end{array} \right]
\end{array}
\tag{1}
$$

Extensive & Strategic Form of the Finite Game Model

Extensive Form:

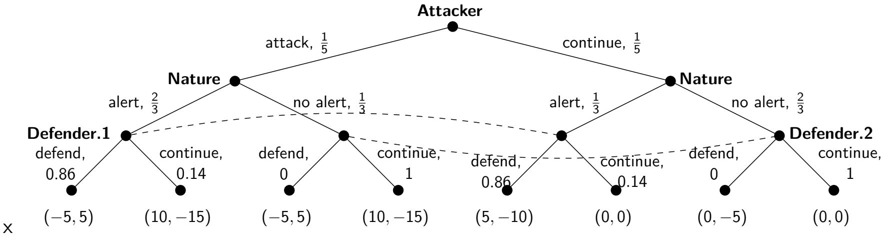


Strategic Form:

$$\begin{array}{c} \\ A \\ C \end{array} \begin{array}{c} \begin{matrix} DC & DD & CD & CC \end{matrix} \\ \left[\begin{matrix} 0, \underline{-\frac{5}{3}} & -5, \underline{5} & 5, -\frac{25}{3} & \underline{10}, -15 \\ \underline{\frac{5}{3}}, -\frac{10}{3} & \underline{\frac{5}{3}}, -\frac{20}{3} & 0, \underline{-\frac{10}{3}} & 0, \underline{0} \end{matrix} \right] \end{array} \tag{2}$$

$\Longrightarrow$ **No pure Nash equilibrium**

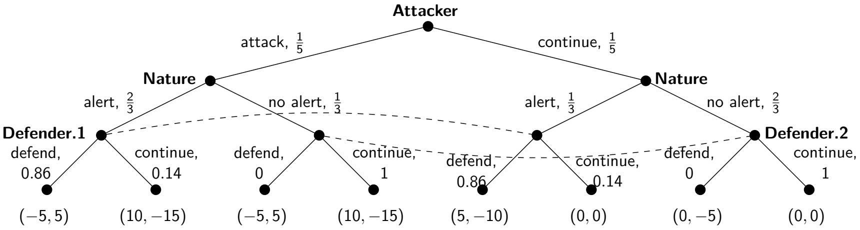
Unique Mixed Nash Equilibrium Computed using Lemke-Howson's Algorithm



Mixed Nash equilibrium $(s_1^*, s_2^*) \in \Delta(A_1) \times \Delta(A_2)$:

- $s_1^*(\text{Attack}) = \frac{1}{5}$, $s_1^*(\text{Continue}) = \frac{4}{5}$
- $s_2^*(\text{Defend}|\text{Alert}) = 0.86$, $s_2^*(\text{Continue}|\text{Alert}) = 0.14$, $s_2^*(\text{Defend}|\text{No Alert}) = 0$, $s_2^*(\text{Continue}|\text{No Alert}) = 1$.

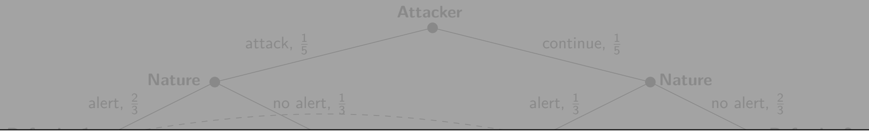
Limitations of the Finite Game Model



Limitations:

- Hard to analyze for large systems (scalability)
- Hard to define all of the parameters in the model
- Equilibria depend greatly on the payoff-values, are they realistic?
- Does mixed equilibria make sense?

Limitations of the Finite Game Model



Alternative model: continuous-kernel game

Limitations:

- Hard to analyze for large systems (scalability)
- Hard to define all of the parameters in the model

Continuous-Kernel Game Model (1/3)

- **Attacker Action Space**: $A_1 = \mathbb{R}_+^{A_{max}}$ instead of $A_1 = \{a_1, \ldots, A_{max}\}$
- **Defender Action Space**: $A_2 = \mathbb{R}_+^{D_{max}}$ instead of $A_2 = \{a_1, \ldots, D_{max}\}$
- **Nature/(Sensor Network) Action Space**:
  - $A_3 = \mathbb{R}_+^{A_{max} \times A_{max}}$ instead of $A_3 = \{\text{alert}, \text{no alert}\} \times A_2 = \bar{\boldsymbol{P}}$
  - $\bar{\boldsymbol{P}}_{ij}$ represent the alert-weight that nature put on attack j when attack i occurred.
  - $\bar{\boldsymbol{P}} = I$ if the sensors are perfect.
  - Attack detection metric: $dq(i) = \frac{\bar{\boldsymbol{P}}_{ii}}{\sum_j^{A_{max}} \bar{\boldsymbol{P}}_{ij}}$
  - For notational convenience, define:

$$\boldsymbol{P} = \begin{cases} p_{i,j} = -\bar{p}_{i,j} & \text{if } i = j \\ p_{i,j} = \bar{p}_{i,j} & \text{otherwise} \end{cases} \quad (3)$$

Continuous-Kernel Game Model (1/3)

- **Attacker Action Space**: $A_1 = \mathbb{R}_+^{A_{max}}$ instead of $A_1 = \{a_1, \ldots, A_{max}\}$
- **Defender Action Space**: $A_2 = \mathbb{R}_+^{D_{max}}$ instead of $A_2 = \{a_1, \ldots, D_{max}\}$
- **Nature/(Sensor Network) Action Space**:
  - $A_3 = \mathbb{R}_+^{A_{max} \times A_{max}}$ instead of $A_3 = \{\text{alert}, \text{no alert}\} \times A_2 = \bar{\boldsymbol{P}}$
  - $\bar{\boldsymbol{P}}_{ij}$ represent the alert-weight that nature put on attack j when attack i occurred.
  - $\bar{\boldsymbol{P}} = I$ if the sensors are perfect.
  - Attack detection metric: $dq(i) = \frac{\bar{\boldsymbol{P}}_{ii}}{\sum_j^{A_{max}} \bar{\boldsymbol{P}}_{ij}}$
  - For notational convenience, define:

$$\boldsymbol{P} = \begin{cases} p_{i,j} = -\bar{p}_{i,j} & \text{if } i = j \\ p_{i,j} = \bar{p}_{i,j} & \text{otherwise} \end{cases} \tag{4}$$

Continuous-Kernel Game Model (1/3)

- **Attacker Action Space**: $A_1 = \mathbb{R}_+^{A_{max}}$ instead of $A_1 = \{a_1, \ldots, A_{max}\}$, pure strategy $\boldsymbol{a}^A \in A_1$

- **Defender Action Space**: $A_2 = \mathbb{R}_+^{D_{max}}$ instead of $A_2 = \{a_1, \ldots, D_{max}\}$, pure strategy $\boldsymbol{a}^D \in A_2$

- **Nature/(Sensor Network) Action Space**:
  - $A_3 = \mathbb{R}_+^{A_{max} \times A_{max}}$ instead of $A_3 = \{\text{alert}, \text{no alert}\} \times A_2 = \bar{\boldsymbol{P}}$
  - $\bar{\boldsymbol{P}}_{ij}$ represent the alert-weight that nature put on attack j when attack i occurred.
  - $\bar{\boldsymbol{P}} = I$ if the sensors are perfect.
  - Attack detection metric: $dq(i) = \frac{\bar{\boldsymbol{P}}_{ii}}{\sum_j^{A_{max}} \bar{\boldsymbol{P}}_{ij}}$
  - For notational convenience, define:

$$\boldsymbol{P} = \begin{cases} p_{i,j} = -\bar{p}_{i,j} & \text{if } i = j \\ p_{i,j} = \bar{p}_{i,j} & \text{otherwise} \end{cases} \quad (5)$$

Continuous-Kernel Game Model (2/3)

- **Defender cost of being attacked:** $\boldsymbol{c}^D \in \mathbb{R}^{A_{max}}$
- **Attacker gain of successful attack:** $\boldsymbol{c}^A \in \mathbb{R}^{A_{max}}$

- **Vulnerability matrix:** $\boldsymbol{Q} \in \mathbb{R}^{A_{max} \times A_{max}}$ diagonal matrix that models degree of vulnerability per attack
- **Defense matrix:** $\bar{\boldsymbol{Q}} \in \{0, 1\}^{A_{max} \times D_{max}}$ where $\bar{\boldsymbol{Q}}_{i,j} = 1$ if defense detects the attack and 0 otherwise.

- **Cost of defender actions:** $\alpha \in \mathbb{R}_+^{D_{max}}$
- **Cost of attacker actions:** $\beta \in \mathbb{R}_+^{A_{max}}$

- **False alarm weight:** γ, defines how much to penalize false alarms

Continuous-Kernel Game Model (2/3)

- **Defender cost of being attacked:** $\boldsymbol{c}^D \in \mathbb{R}^{A_{max}}$
- **Attacker gain of successful attack:** $\boldsymbol{c}^A \in \mathbb{R}^{A_{max}}$

- **Vulnerability matrix:** $\boldsymbol{Q} \in \mathbb{R}^{A_{max} \times A_{max}}$ diagonal matrix that models degree of vulnerability per attack
- **Defense matrix:** $\bar{\boldsymbol{Q}} \in \{0, 1\}^{A_{max} \times D_{max}}$ where $\bar{\boldsymbol{Q}}_{i,j} = 1$ if defense detects the attack and 0 otherwise.

- **Cost of defender actions:** $\alpha \in \mathbb{R}_+^{D_{max}}$
- **Cost of attacker actions:** $\beta \in \mathbb{R}_+^{A_{max}}$

- **False alarm weight:** γ, defines how much to penalize false alarms

Continuous-Kernel Game Model (2/3)

- **Defender cost of being attacked:** $\boldsymbol{c}^D \in \mathbb{R}^{A_{max}}$
- **Attacker gain of successful attack:** $\boldsymbol{c}^A \in \mathbb{R}^{A_{max}}$

- **Vulnerability matrix:** $\boldsymbol{Q} \in \mathbb{R}^{A_{max} \times A_{max}}$ diagonal matrix that models degree of vulnerability per attack
- **Defense matrix:** $\bar{\boldsymbol{Q}} \in \{0,1\}^{A_{max} \times D_{max}}$ where $\bar{\boldsymbol{Q}}_{i,j} = 1$ if defense detects the attack and 0 otherwise.

- **Cost of defender actions:** $\alpha \in \mathbb{R}_+^{D_{max}}$
- **Cost of attacker actions:** $\beta \in \mathbb{R}_+^{A_{max}}$

- **False alarm weight:** γ, defines how much to penalize false alarms

Continuous-Kernel Game Model (2/3)

- **Defender cost of being attacked:** $\boldsymbol{c}^D \in \mathbb{R}^{A_{max}}$
- **Attacker gain of successful attack:** $\boldsymbol{c}^A \in \mathbb{R}^{A_{max}}$

- **Vulnerability matrix:** $\boldsymbol{Q} \in \mathbb{R}^{A_{max} \times A_{max}}$ diagonal matrix that models degree of vulnerability per attack
- **Defense matrix:** $\bar{\boldsymbol{Q}} \in \{0, 1\}^{A_{max} \times D_{max}}$ where $\bar{\boldsymbol{Q}}_{i,j} = 1$ if defense detects the attack and 0 otherwise.

- **Cost of defender actions:** $\alpha \in \mathbb{R}_+^{D_{max}}$
- **Cost of attacker actions:** $\beta \in \mathbb{R}_+^{A_{max}}$

- **False alarm weight:** γ, defines how much to penalize false alarms

Continuous-Kernel Game Model (3/3)

- **Objective:** Minimize costs rather than maximize utilities

- **Defender Cost Function:**

$$J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P}) = \underbrace{\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D}_{\text{false alarm}} + \underbrace{(\boldsymbol{a}^D)^T diag(\alpha)\boldsymbol{a}^D}_{\text{cost of defense}} + \underbrace{\boldsymbol{c}^D(\boldsymbol{Q}\boldsymbol{a}^A - \bar{\boldsymbol{Q}}\boldsymbol{a}^D)}_{\text{cost of attack}} \tag{6}$$

- **Attacker Cost Function:**

$$J^A(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P}) = -\underbrace{\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D}_{\text{detected}} + \underbrace{(\boldsymbol{a}^A)^T diag(\beta)\boldsymbol{a}^A}_{\text{cost of attack}} + \underbrace{\boldsymbol{c}^A(\bar{\boldsymbol{Q}}\boldsymbol{a}^D - \boldsymbol{Q}\boldsymbol{a}^A)}_{\text{gain of attack}} \tag{7}$$

- **Summary:** Continuous-kernel general-sum game with strictly convex cost functions.

Continuous-Kernel Game Model (3/3)

- **Objective:** Minimize costs rather than maximize utilities

- **Defender Cost Function:**

$$J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P}) = \underbrace{\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D}_{\text{false alarm}} + \underbrace{(\boldsymbol{a}^D)^T diag(\alpha)\boldsymbol{a}^D}_{\text{cost of defense}} + \underbrace{\boldsymbol{c}^D(\boldsymbol{Q}\boldsymbol{a}^A - \bar{\boldsymbol{Q}}\boldsymbol{a}^D)}_{\text{cost of attack}} \quad (8)$$

- **Attacker Cost Function:**

$$J^A(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P}) = -\underbrace{\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D}_{\text{detected}} + \underbrace{(\boldsymbol{a}^A)^T diag(\beta)\boldsymbol{a}^A}_{\text{cost of attack}} + \underbrace{\boldsymbol{c}^A(\bar{\boldsymbol{Q}}\boldsymbol{a}^D - \boldsymbol{Q}\boldsymbol{a}^A)}_{\text{gain of attack}} \quad (9)$$

- **Summary:** Continuous-kernel general-sum game with strictly convex cost functions.

Continuous-Kernel Game Model (3/3)

- **Objective:** Minimize costs rather than maximize utilities

- **Defender Cost Function:**

$$J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P}) = \underbrace{\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D}_{\text{false alarm}} + \underbrace{(\boldsymbol{a}^D)^T diag(\alpha)\boldsymbol{a}^D}_{\text{cost of defense}} + \underbrace{\boldsymbol{c}^D(\boldsymbol{Q}\boldsymbol{a}^A - \bar{\boldsymbol{Q}}\boldsymbol{a}^D)}_{\text{cost of attack}} \tag{10}$$

- **Attacker Cost Function:**

$$J^A(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P}) = -\underbrace{\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D}_{\text{detected}} + \underbrace{(\boldsymbol{a}^A)^T diag(\beta)\boldsymbol{a}^A}_{\text{cost of attack}} + \underbrace{\boldsymbol{c}^A(\bar{\boldsymbol{Q}}\boldsymbol{a}^D - \boldsymbol{Q}\boldsymbol{a}^A)}_{\text{gain of attack}} \tag{11}$$

- **Summary:** Continuous-kernel general-sum game with strictly convex cost functions.

Continuous-Kernel Game Model (3/3)

- **Objective:** Minimize costs rather than maximize utilities

- **Defender Cost Function:**

$$J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P}) = \underbrace{\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D}_{\text{false alarm}} + \underbrace{(\boldsymbol{a}^D)^T diag(\alpha)\boldsymbol{a}^D}_{\text{cost of defense}} + \underbrace{\boldsymbol{c}^D(\boldsymbol{Q}\boldsymbol{a}^A - \bar{\boldsymbol{Q}}\boldsymbol{a}^D)}_{\text{cost of attack}} \quad (12)$$

- **Attacker Cost Function:**

$$J^A(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P}) = -\underbrace{\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D}_{\text{detected}} + \underbrace{(\boldsymbol{a}^A)^T diag(\beta)\boldsymbol{a}^A}_{\text{cost of attack}} + \underbrace{\boldsymbol{c}^A(\bar{\boldsymbol{Q}}\boldsymbol{a}^D - \boldsymbol{Q}\boldsymbol{a}^A)}_{\text{gain of attack}} \quad (13)$$

- **Summary:** Continuous-kernel general-sum game with convex cost functions.

Equilibrium Analysis (1/3)

- Since J^A, J^D are strictly convex, the **best-response correspondences** are obtained from the first order conditions:
-

$$\nabla_{\boldsymbol{a}^D}(J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})) = 0 \tag{14}$$
$$\nabla_{\boldsymbol{a}^D}(\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D + (\boldsymbol{a}^D)^T diag(\alpha)\boldsymbol{a}^D + \boldsymbol{c}^D(\boldsymbol{Q}\boldsymbol{a}^A - \bar{\boldsymbol{Q}}\boldsymbol{a}^D)) = 0$$
$$\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}} + (\boldsymbol{a}^D)^T(2diag(\alpha)) - \boldsymbol{c}^D\bar{\boldsymbol{Q}} = 0$$
$$\implies Br_D(\boldsymbol{a}^A, \boldsymbol{P}) = \{\boldsymbol{c}^D\bar{\boldsymbol{Q}}(2diag(\alpha))^{-1} - \gamma(2diag(\alpha))^{-1}\bar{\boldsymbol{Q}}^T\boldsymbol{P}^T\boldsymbol{a}^A$$

and, analogously for the attacker:

$$\nabla_{\boldsymbol{a}^A}(J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})) = 0 \tag{15}$$
$$-\gamma\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D + (\boldsymbol{a}^A)^T(2diag(\beta)) - \boldsymbol{c}^A\boldsymbol{Q} = 0$$
$$\boldsymbol{a}^A = (2diag(\beta))^{-1}\boldsymbol{c}^A\boldsymbol{Q} + \gamma(2diag(\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D$$
$$\implies Br_A(\boldsymbol{a}^D, \boldsymbol{P}) = \{(2diag(\beta))^{-1}\boldsymbol{c}^A\boldsymbol{Q} + \gamma(2diag(\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D\}$$

Equilibrium Analysis (1/3)

- Since J^A, J^D are strictly convex, the **best-response correspondences** are obtained from the first order conditions:

$$
\begin{aligned}
&\nabla_{\boldsymbol{a}^D}(J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})) = 0 \qquad (16)\\
&\nabla_{\boldsymbol{a}^D}(\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D + (\boldsymbol{a}^D)^T diag(\alpha)\boldsymbol{a}^D + \boldsymbol{c}^D(\boldsymbol{Q}\boldsymbol{a}^A - \bar{\boldsymbol{Q}}\boldsymbol{a}^D)) = 0\\
&\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}} + (\boldsymbol{a}^D)^T(2diag(\alpha)) - \boldsymbol{c}^D\bar{\boldsymbol{Q}} = 0\\
&\implies Br_D(\boldsymbol{a}^A, \boldsymbol{P}) = \{\boldsymbol{c}^D\bar{\boldsymbol{Q}}(2diag(\alpha))^{-1} - \gamma(2diag(\alpha))^{-1}\bar{\boldsymbol{Q}}^T\boldsymbol{P}^T\boldsymbol{a}^A\}
\end{aligned}
$$

- and, analogously for the attacker:

$$
\begin{aligned}
&\nabla_{\boldsymbol{a}^A}(J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})) = 0 \qquad (17)\\
&-\gamma\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D + (\boldsymbol{a}^A)^T(2diag(\beta)) - \boldsymbol{c}^A\boldsymbol{Q} = 0\\
&\boldsymbol{a}^A = (2diag(\beta))^{-1}\boldsymbol{c}^A\boldsymbol{Q} + \gamma(2diag(\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D\\
&\implies Br_A(\boldsymbol{a}^D, \boldsymbol{P}) = \{(2diag(\beta))^{-1}\boldsymbol{c}^A\boldsymbol{Q} + \gamma(2diag(\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D\}
\end{aligned}
$$

Equilibrium Analysis (1/3)

- Since J^A, J^D are strictly convex, the **best-response correspondences** are obtained from the first order conditions:

$$\nabla_{\boldsymbol{a}^D}(J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})) = 0 \tag{18}$$
$$\nabla_{\boldsymbol{a}^D}(\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D + (\boldsymbol{a}^D)^T diag(\alpha)\boldsymbol{a}^D + \boldsymbol{c}^D(\boldsymbol{Q}\boldsymbol{a}^A - \bar{\boldsymbol{Q}}\boldsymbol{a}^D)) = 0$$
$$\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}} + (\boldsymbol{a}^D)^T(2diag(\alpha)) - \boldsymbol{c}^D\bar{\boldsymbol{Q}} = 0$$
$$\implies Br_D(\boldsymbol{a}^A, \boldsymbol{P}) = \{\boldsymbol{c}^D\bar{\boldsymbol{Q}}(2diag(\alpha))^{-1} - \gamma(2diag(\alpha))^{-1}\bar{\boldsymbol{Q}}^T\boldsymbol{P}^T\boldsymbol{a}^A\}$$

and, analogously for the attacker:

$$\nabla_{\boldsymbol{a}^A}(J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})) = 0 \tag{19}$$
$$-\gamma\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D + (\boldsymbol{a}^A)^T(2diag(\beta)) - \boldsymbol{c}^A\boldsymbol{Q} = 0$$
$$\boldsymbol{a}^A = (2diag(\beta))^{-1}\boldsymbol{c}^A\boldsymbol{Q} + \gamma(2diag(\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D$$
$$\implies Br_A(\boldsymbol{a}^D, \boldsymbol{P}) = \{(2diag(\beta))^{-1}\boldsymbol{c}^A\boldsymbol{Q} + \gamma(2diag(\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D\}$$

Equilibrium Analysis (2/3)

- For notational convenience, define

$$\theta^D(\boldsymbol{c}^D\bar{\boldsymbol{Q}}, \alpha) \triangleq [(\boldsymbol{c}^D\bar{\boldsymbol{Q}})_1/(2\alpha_1), \ldots, (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_{D_{max}}/(2\alpha_{D_{max}})]$$
$$\theta^A(\boldsymbol{c}^A\boldsymbol{Q}, \beta) \triangleq [(\boldsymbol{c}^A\boldsymbol{Q})_1/(2\beta_1), \ldots, (\boldsymbol{c}^A\boldsymbol{Q})_{A_{max}}/(2\beta_{A_{max}})]$$

- Then we can write the best response functions as:

$$Br_D(\boldsymbol{a}^A, \boldsymbol{P}) = [\theta^D - \gamma(diag(2\alpha))^{-1}\bar{\boldsymbol{Q}}^T\boldsymbol{P}^T\boldsymbol{a}^A]^+$$
$$Br_A(\boldsymbol{a}^D, \boldsymbol{P}) = [\theta^A + \gamma(diag(2\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D]^+$$

- The set of Nash equilibria is

$$\{(\boldsymbol{a}^A, \boldsymbol{a}^D) \quad | \quad \boldsymbol{a}^A \in Br_A(\boldsymbol{a}^D, \boldsymbol{P}), \boldsymbol{a}^D \in Br_D(\boldsymbol{a}^A, \boldsymbol{P})\} \quad (20)$$

- How large is this set? What do the elements of this set look like?

Equilibrium Analysis (3/3)

- For notational convenience, define

$$\theta^D(\boldsymbol{c}^D\bar{\boldsymbol{Q}}, \alpha) \triangleq [(\boldsymbol{c}^D\bar{\boldsymbol{Q}})_1/(2\alpha_1), \dots, (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_{D_{max}}/(2\alpha_{D_{max}})]$$
$$\theta^A(\boldsymbol{c}^A\boldsymbol{Q}, \beta) \triangleq [(\boldsymbol{c}^A\boldsymbol{Q})_1/(2\beta_1), \dots, (\boldsymbol{c}^A\boldsymbol{Q})_{A_{max}}/(2\beta_{A_{max}})]$$

- Then we can write the best response functions as:

$$Br_D(\boldsymbol{a}^A, \boldsymbol{P}) = [\theta^D - \gamma(diag(2\alpha))^{-1}\bar{\boldsymbol{Q}}^T\boldsymbol{P}^T\boldsymbol{a}^A]^+$$
$$Br_A(\boldsymbol{a}^D, \boldsymbol{P}) = [\theta^A + \gamma(diag(2\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D]^+$$

- The set of Nash equilibria is

$$\{(\boldsymbol{a}^A, \boldsymbol{a}^D) \quad | \quad \boldsymbol{a}^A \in Br_A(\boldsymbol{a}^D, \boldsymbol{P}), \boldsymbol{a}^D \in Br_D(\boldsymbol{a}^A, \boldsymbol{P})\} \quad (21)$$

- How large is this set? What do the elements of this set look like?

Main Contribution of the Paper

Theorem

There exists a unique NE. Further, if:

$$\gamma < \min\left[\frac{\min_i \theta^D}{\max_i (diag(2\alpha))^{-1} \bar{\boldsymbol{Q}}^T \boldsymbol{P}^T \theta^A}, \frac{\max_i \theta^A}{\max_i (diag(2\beta))^{-1} (-\boldsymbol{P}) \bar{\boldsymbol{Q}} \theta^D}\right] \quad (22)$$

Then the unique NE $(\boldsymbol{a}^{D*}, \boldsymbol{a}^{A*})$ *satisfy* $\boldsymbol{a}^{D,*} > 0$ *and* $\boldsymbol{a}^{A,*} > 0$ *and is given by:*

$$\boldsymbol{a}^{A*} = (I + Z)^{-1}[\theta^A + \gamma(diag(2\beta))^{-1} \boldsymbol{P} \bar{\boldsymbol{Q}} \theta^D] \quad (23)$$

$$\boldsymbol{a}^{D*} = (I + \bar{Z})^{-1}[\theta^D - \gamma(diag(2\alpha))^{-1} \bar{\boldsymbol{Q}}^T \boldsymbol{P}^T \theta^A] \quad (24)$$

where $Z \triangleq \gamma^2 (diag(2\beta))^{-1} \boldsymbol{P} \bar{\boldsymbol{Q}} (diag(2\alpha))^{-1} \bar{\boldsymbol{Q}}^T \boldsymbol{P}^T$ *and*
$\bar{Z} \triangleq \gamma^2 (diag(2\alpha))^{-1} \bar{\boldsymbol{Q}}^T \boldsymbol{P}^T (diag(2\beta))^{-1} \boldsymbol{P} \bar{\boldsymbol{Q}}$.

Rosen's Existence Theorem[2][3]

Theorem (Pure Nash Equilibrium Existence for Continuous-Kernel Games)

For each player $i \in N$, let A_i be a compact and convex subset of a finite-dimensional Euclidean space, and the cost functional $J^i : A_1 \times \ldots \times A_N \to \mathbb{R}$ be jointly continuous in all its arguments and strictly convex in a_i for every $a_j \in A_j, j \in N, j \neq i$. Then, the associated N-person nonzero-sum game admits a Nash equilibrium in pure strategies.

---

[2] T. Başar and G.J. Olsder. *Dynamic Noncooperative Game Theory*. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics, 1999. ISBN: 9780898714296. URL: https://books.google.se/books?id=k1oF5AxmJlYC.

[3] J. B. Rosen. "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games". In: *Econometrica* 33.3 (1965), pp. 520–534. ISSN: 00129682, 14680262. URL: http://www.jstor.org/stable/1911749.

Proof of Theorem 1, Existence

- **Existence of Pure NE**:
  - A^A, A^D are convex subsets of a Euclidean space
  - J^A, J^D are jointly continuous in all their arguments and strictly convex in $\boldsymbol{a}^A, \boldsymbol{a}^D$ respectively,
  - A^A, A^D are not compact.
  - However, $J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})$ and $J^A(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})$ grow unbounded as $|\boldsymbol{a}| \to \infty$
  - $\implies$ by Rosen's existence theorem, the game has a pure Nash equilibrium.

Proof of Theorem 1, Existence

- **Existence of Pure NE**:
  - A^A, A^D are convex subsets of a Euclidean space
  - J^A, J^D are jointly continuous in all their arguments and strictly convex in $\boldsymbol{a}^A, \boldsymbol{a}^D$ respectively,
  - A^A, A^D are not compact.
  - However, $J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})$ and $J^A(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})$ grow unbounded as $|\boldsymbol{a}| \to \infty$
  - $\implies$ by Rosen's existence theorem, the game has a pure Nash equilibrium.

Proof of Theorem 1, Existence

- **Existence of Pure NE**:
  - A^A, A^D are convex subsets of a Euclidean space
  - J^A, J^D are jointly continuous in all their arguments and strictly convex in $\boldsymbol{a}^A, \boldsymbol{a}^D$ respectively,
  - A^A, A^D are not compact.
  - However, $J^D(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})$ and $J^A(\boldsymbol{a}^A, \boldsymbol{a}^D, \boldsymbol{P})$ grow unbounded as $|\boldsymbol{a}| \to \infty$
  - $\implies$ by Rosen's existence theorem, the game has a pure Nash equilibrium.

Rosen's Uniqueness Theorem (1/3) - Pseudo-gradient[4]

Definition (Pseudo-Gradient $g(a)$ and Pseudo-Gradient Operator $\bar{\nabla}$)

Let $\bar{\nabla}$ be the pseudo-gradient operator, defined through its application on the cost vector J as:

$$\bar{\nabla} J \triangleq \begin{bmatrix} \frac{\partial J_1(a)}{\partial a_1} \\ \frac{\partial J_2(a)}{\partial a_2} \\ \vdots \\ \frac{\partial J_{|N|}(a)}{\partial a_{|N|}} \end{bmatrix} = g(a) \tag{25}$$

[4] J. B. Rosen. "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games". In: *Econometrica* 33.3 (1965), pp. 520–534. ISSN: 00129682, 14680262. URL: http://www.jstor.org/stable/1911749.

Proof Preliminaries (2/3) - Pseudo-Hessian[5]

Definition (Pseudo-Hessian)

Let $G(a)$ be the Jacobian of the pseudo-gradient $g(a)$ with respect to a (also called pseudo-Hessian):

$$G(a) \triangleq \begin{bmatrix} \frac{\partial^2 J_1(a)}{\partial a_1^2} & \frac{\partial^2 J_1(a)}{\partial a_1 \partial a_2} & \dots & \frac{\partial^2 J_1(a)}{\partial a_1 \partial a_{|N|}} \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 J_{|N|}(a)}{\partial a_{|N|} \partial a_1} & \frac{\partial^2 J_{|N|}(a)}{\partial a_{|N|} \partial a_2} & \dots & \frac{\partial^2 J_{|N|}(a)}{\partial a_{|N|}^2} \end{bmatrix} \tag{26}$$

Definition (Symmetrized Pseudo-Hessian)

Let $G(a)$ be the Jacobian of the pseudo-gradient $g(a)$ with respect to a, i.e. the pseudo-Hessian, then the symmetrized pseudo-hessian is defined as:

$$\mathcal{G}(a) \triangleq G(a) + G(a)^T \tag{27}$$

Proof Preliminaries (3/3) - Rosen's Uniqueness Theorem[6]

Theorem (Unique Pure Nash Equilibrium Existence for Continuous-Kernel Games)

If the symmetrized pseudo-Hessian $\mathcal{G}(a)$ is positive definite, the pure equilibrium of a continuous-kernel game with strictly convex cost functions is unique.

[6] J. B. Rosen. "Existence and Uniqueness of Equilibrium Points for Concave N-Person Games". In: *Econometrica* 33.3 (1965), pp. 520–534. ISSN: 00129682, 14680262. URL: http://www.jstor.org/stable/1911749.

Proof of Theorem 1, Uniqueness

The pseudo-gradient is:

$$\bar{\nabla} J(\boldsymbol{a}) = \begin{bmatrix} (\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}})_1 + (\boldsymbol{a}^D)^T(2\alpha_1) - (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_1 & \dots & (\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}})_{D_{max}} + (\boldsymbol{a}^D)^T(2\alpha_{D_{max}}) - (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_{D_{max}} \\ -(\gamma\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D)_1 + (\boldsymbol{a}^A)^T(2\beta_1) - (\boldsymbol{c}^A\boldsymbol{Q})_1 & \dots & -(\gamma\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D)_{A_{max}} + (\boldsymbol{a}^A)^T(2\beta_{A_{max}}) - (\boldsymbol{c}^A\boldsymbol{Q})_{A_{max}} \end{bmatrix} \tag{28}$$

Proof of Theorem 1, Uniqueness

The pseudo-gradient is:

$$\bar{\nabla} J(\boldsymbol{a}) = \begin{bmatrix} (\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}})_1 + (\boldsymbol{a}^D)^T(2\alpha_1) - (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_1 & \dots & (\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}})_{D_{max}} + (\boldsymbol{a}^D)^T(2\alpha_{D_{max}}) - (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_{D_{max}} \\ -(\gamma \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D)_1 + (\boldsymbol{a}^A)^T(2\beta_1) - (\boldsymbol{c}^A\boldsymbol{Q})_1 & \dots & -(\gamma \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D)_{A_{max}} + (\boldsymbol{a}^A)^T(2\beta_{A_{max}}) - (\boldsymbol{c}^A\boldsymbol{Q})_{A_{max}} \end{bmatrix} \tag{29}$$

The pseudo-hessian is:

$$G(\boldsymbol{a}) = \left[\begin{array}{ccc|ccc} 2\alpha_1 & 0 & 0 & & & \\ 0 & \ddots & 0 & & \gamma\boldsymbol{P}\bar{\boldsymbol{Q}} & \\ 0 & 0 & 2\alpha_{D_{max}} & & & \\ \hline & & & 2\beta_1 & 0 & 0 \\ & -\gamma\boldsymbol{P}\bar{\boldsymbol{Q}} & & 0 & \ddots & 0 \\ & & & 0 & 0 & 2\beta_{A_{max}} \end{array}\right] \tag{30}$$

Proof of Theorem 1, Uniqueness

The pseudo-gradient is:

$$\bar{\nabla} J(\boldsymbol{a}) = \begin{bmatrix} (\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}})_1 + (\boldsymbol{a}^D)^T(2\alpha_1) - (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_1 & \dots & (\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}})_{D_{max}} + (\boldsymbol{a}^D)^T(2\alpha_{D_{max}}) - (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_{D_{max}} \\ -(\gamma \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D)_1 + (\boldsymbol{a}^A)^T(2\beta_1) - (\boldsymbol{c}^A\boldsymbol{Q})_1 & \dots & -(\gamma \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D)_{A_{max}} + (\boldsymbol{a}^A)^T(2\beta_{A_{max}}) - (\boldsymbol{c}^A\boldsymbol{Q})_{A_{max}} \end{bmatrix} \tag{31}$$

The pseudo-hessian is:

$$G(\boldsymbol{a}) = \left[\begin{array}{ccc|ccc} 2\alpha_1 & 0 & 0 & & & \\ 0 & \ddots & 0 & & \gamma\boldsymbol{P}\bar{\boldsymbol{Q}} & \\ 0 & 0 & 2\alpha_{D_{max}} & & & \\ \hline & & & 2\beta_1 & 0 & 0 \\ & -\gamma\boldsymbol{P}\bar{\boldsymbol{Q}} & & 0 & \ddots & 0 \\ & & & 0 & 0 & 2\beta_{A_{max}} \end{array}\right] \tag{32}$$

Clearly, $\mathcal{G}(\boldsymbol{a}) = G(\boldsymbol{a}) + G(\boldsymbol{a})^T = 4diag([\alpha, \beta]^T)$, which is positive definite.

Proof of Theorem 1, Uniqueness

The pseudo-gradient is:

$$\bar{\nabla} J(\boldsymbol{a}) = \begin{bmatrix} (\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}})_1 + (\boldsymbol{a}^D)^T(2\alpha_1) - (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_1 & \dots & (\gamma(\boldsymbol{a}^A)^T \boldsymbol{P}\bar{\boldsymbol{Q}})_{D_{max}} + (\boldsymbol{a}^D)^T(2\alpha_{D_{max}}) - (\boldsymbol{c}^D\bar{\boldsymbol{Q}})_{D_{max}} \\ -(\gamma \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D)_1 + (\boldsymbol{a}^A)^T(2\beta_1) - (\boldsymbol{c}^A\boldsymbol{Q})_1 & \dots & -(\gamma \boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D)_{A_{max}} + (\boldsymbol{a}^A)^T(2\beta_{A_{max}}) - (\boldsymbol{c}^A\boldsymbol{Q})_{A_{max}} \end{bmatrix} \tag{33}$$

The pseudo-hessian is:

$$G(\boldsymbol{a}) = \left[\begin{array}{ccc|ccc} 2\alpha_1 & 0 & 0 & & & \\ 0 & \ddots & 0 & & \gamma\boldsymbol{P}\bar{\boldsymbol{Q}} & \\ 0 & 0 & 2\alpha_{D_{max}} & & & \\ \text{—} & \text{—} & \text{—} & \text{—} & \text{—} & \text{—} \\ & & & 2\beta_1 & 0 & 0 \\ & -\gamma\boldsymbol{P}\bar{\boldsymbol{Q}} & & 0 & \ddots & 0 \\ z & & & 0 & 0 & 2\beta_{A_{max}} \end{array}\right] \tag{34}$$

Clearly, $\mathcal{G}(\boldsymbol{a}) = G(\boldsymbol{a}) + G(\boldsymbol{a})^T = 4diag([\alpha, \beta]^T)$, which is positive definite. Thus, by Rosen's Uniqueness theorem, **the NE is unique**

Proof of Theorem 1, Analytical Characterization of NE

Recall the Best Response Functions:

$$Br_D(\boldsymbol{a}^A, \boldsymbol{P}) = [\theta^D - \gamma(diag(2\alpha))^{-1}\bar{\boldsymbol{Q}}^T \boldsymbol{P}^T \boldsymbol{a}^A]^+$$
$$Br_A(\boldsymbol{a}^D, \boldsymbol{P}) = [\theta^A + \gamma(diag(2\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D]^+$$

Proof of Theorem 1, Analytical Characterization of NE

Recall the Best Response Functions:

$$Br_D(\boldsymbol{a}^A, \boldsymbol{P}) = [\theta^D - \gamma(diag(2\alpha))^{-1}\bar{\boldsymbol{Q}}^T\boldsymbol{P}^T\boldsymbol{a}^A]^+$$
$$Br_A(\boldsymbol{a}^D, \boldsymbol{P}) = [\theta^A + \gamma(diag(2\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D]^+$$

Substitute $\boldsymbol{a}^D$ in $Br_A(\boldsymbol{a}^D, \boldsymbol{P})$ with $Br_D(\boldsymbol{a}^A, \boldsymbol{P})$, we then obtain the fixed point equation:

$$\begin{aligned}\boldsymbol{a}^{A*} &= Br_A(Br_D(\boldsymbol{a}^{A*}, \boldsymbol{P}), \boldsymbol{P}) \\ &= [\theta^A + \gamma(diag(2\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}[\theta^D - \gamma(diag(2\alpha))^{-1}\bar{\boldsymbol{Q}}^T\boldsymbol{P}^T\boldsymbol{a}^{A*}]^+]^+\end{aligned}$$

Proof of Theorem 1, Analytical Characterization of NE

Recall the Best Response Functions:

$$Br_D(\boldsymbol{a}^A, \boldsymbol{P}) = [\theta^D - \gamma(diag(2\alpha))^{-1}\bar{\boldsymbol{Q}}^T \boldsymbol{P}^T \boldsymbol{a}^A]^+$$
$$Br_A(\boldsymbol{a}^D, \boldsymbol{P}) = [\theta^A + \gamma(diag(2\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\boldsymbol{a}^D]^+$$

Substitute $\boldsymbol{a}^D$ in $Br_A(\boldsymbol{a}^D, \boldsymbol{P})$ with $Br_D(\boldsymbol{a}^A, \boldsymbol{P})$, we then obtain the fixed point equation:

$$\begin{aligned}\boldsymbol{a}^{A*} &= Br_A(Br_D(\boldsymbol{a}^{A*}, \boldsymbol{P}), \boldsymbol{P}) \\ &= [\theta^A + \gamma(diag(2\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}[\theta^D - \gamma(diag(2\alpha))^{-1}\bar{\boldsymbol{Q}}^T \boldsymbol{P}^T \boldsymbol{a}^{A*}]^+]^+\end{aligned}$$

Solving for $\boldsymbol{a}^{A*}$ and similarly for $\boldsymbol{a}^{D*}$ yields:

$$\boldsymbol{a}^{A*} = (I + Z)^{-1}[\theta^A + \gamma(diag(2\beta))^{-1}\boldsymbol{P}\bar{\boldsymbol{Q}}\theta^D] \quad (35)$$
$$\boldsymbol{a}^{D*} = (I + \bar{Z})^{-1}[\theta^D - \gamma(diag(2\alpha))^{-1}\bar{\boldsymbol{Q}}^T \boldsymbol{P}^T \theta^A] \quad (36)$$

Conclusion

- **Topic:**
  - The paper provides a game theoretic analysis of intrusion detection
- **Contributions:**
  - A finite extensive form non-cooperative game model
  - A infinite continuous-kernel strategic non-cooperative game model
  - Existence and uniqueness proof of NE
  - Repeated game simulation

Discussion

- **General questions/Comments?**
- **Are there other existence/uniqueness theorems that could have been used?**
- **Are cyber attacks continuous?**
  - The continuous-kernel model provide a richer analytical analysis
  - But, does it make sense in practice?
- **Which Model makes most sense:**
  - Finite game model with NE in mixed strategies
  - Infinite game model with NE in pure strategies